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Nilpotent Injectors in Finite Groups All of Whose Local Subgroups Are $\mathscr{N}$-Constrained

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INTRODUCTION

A *nilpotent injector* in a finite group G is any maximal nilpotent subgroup I of G satisfying $d(2, I) = d(2, G)$, where $d(2, X)$ is defined as

$$\max\{|A| \mid A \leqslant X \text{ and } A \text{ is nilpotent of class at most } 2\}.$$

Nilpotent injectors have the property that they contain every nilpotent subgroup that they normalize. A. Bialostocki has conjectured [4] that if G is a finite group then all nilpotent injectors of G are conjugate. He has verified his conjecture for the symmetric and alternating groups. The conjecture is also true for soluble groups. The only property of soluble groups used in proving this is $C(F(G)) \leqslant F(G)$. So define a group G to be *$\mathscr{N}$-constrained* if $C(F(G)) \leqslant F(G)$. Soluble groups are $\mathscr{N}$-constrained.

The definition of nilpotent injector used in this paper is due to Bialostocki and is not the same as the more usual definition given in [11]. As Bialostocki points out in [4], if G is $\mathscr{N}$-constrained then the two definitions yield the same class of subgroups. The reader is referred to the introduction of [4] for the reason why Bialostocki's definition is preferable in the context of proving conjugacy theorems.

The main theorem of this paper is the following

THEOREM. *Let G be a finite group all of whose local subgroups are $\mathscr{N}$-constrained. Then all nilpotent injectors of G are conjugate.*

The proof is elementary and does not use any classification theorem. I will give a brief outline of the proof. Consider a counterexample G and take a nilpotent injector I of G. It is easy to see that I has order divisible by more than one prime, so as local subgroups are $\mathscr{N}$-constrained,

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it is possible to show that I is contained in a unique maximal local subgroup L.

Next we seek a prime p and a subgroup $V \cong Z_p \times Z_p$ with $V \leqslant F(L)$ and V normal in a Sylow p-subgroup of L. If this is not possible then the structure of $F(L)$ is very restricted and since L is $\mathcal{N}$-constrained, the structure of L is restricted and it is easy to derive a contradiction.

If $C_G(v) \leqslant L$ for all $v \in V^{\#}$ then it is easy to show that L contains a Sylow p-subgroup of G and then that all nilpotent injectors are conjugate. Hence we may assume $C_G(v) \nleqslant L$ for some $v \in V^{\#}$. Using $\mathcal{N}$-constraint and the definition of a nilpotent injector, it is possible to restrict vastly the structure of I and L. Again it is possible to show that L contains a Sylow p-subgroup of G and that all nilpotent injectors are conjugate.

1. Notation and Quoted Results

For a group G, a positive integer n and a set of primes π, we fix the following notation.

$A(G)$ = the set of maximal local subgroups of G.

$d(2, G) = \max\{|A| \mid A \leqslant G \text{ and } \mathrm{cl}(A) \leqslant 2\}$.

$\pi d(2, G)$ = the set of prime divisors of $d(2, G)$.

$\mathscr{A}(2, G) = \{A \leqslant G \mid \mathrm{cl}(A) \leqslant 2 \text{ and } |A| = d(2, G)\}$.

$\mathcal{N}\mathscr{I}(G) = (I \leqslant G \mid I \text{ maximal nilpotent and contains a member of } \mathscr{A}(2, G)\}$.

Members of $\mathcal{N}\mathscr{I}(G)$ are called nilpotent injectors of G.

Clearly if $A \leqslant H \leqslant G$ with $A \in \mathscr{A}(2, G)$ then $\mathscr{A}(2, H) \subseteq \mathscr{A}(2, G)$. Also if $I \leqslant H \leqslant G$ and $I \in \mathcal{N}\mathscr{I}(G)$ then $I \in \mathcal{N}\mathscr{I}(H)$.

If A and B are groups then $A \leqslant B$ means A is a subgroup of B, $A \precsim B$ means A is isomorphic to a subgroup of B, and $A \lhd B$ means A is a normal subgroup of B. If n is a natural number then Z_n, D_n, and Q_n denote the cyclic, dihedral and generalized quaternion groups of order n, when they exist.

A group G is *quasicyclic* if G is nilpotent, $O(G)$ is cyclic and $O_2(G)$ is cyclic, dihedral, semidihedral, or generalized quaternion. (Note: dihedral includes $Z_2 \times Z_2$.)

LEMMA 1.1. (i) *If A is quasicyclic then so are all its subgroups and* $O_2(A)/\Phi(O_2(A)) \precsim Z_2 \times Z_2$ [6, *p.* 191, *Theorem* 5.4.3 (ii)].

(ii) *Let A be a nilpotent group. If every abelian normal subgroup of A is cyclic then A is quasicyclic* [6, *p.* 199, *Theorem* 5.4.10 (i)].

(iii) *($P \times Q$ Lemma) Let $P \times Q$ act on a p-group V with P a p-group and Q a p'-group. If* $[C_V(P), Q] = 1$ *then* $[V, Q] = 1$ [6, *p.* 179, *Theorem* 5.3.4].

(iv) *Let a noncyclic abelian p-group A act on a p'-group G. Then* $G = \langle C_G(a) \mid a \in A^{\#} \rangle$ [6, *p.* 225, *Theorem* 6.2.4].

(v) *Let P be a p-group, p an odd prime. If P is noncyclic then so is* $\Omega_1(Z_2(P))$ [8, *p.* 303, *Satz* III.7.5, *and p.* 305, *Satz* III.7.7].

(vi) *Let x and y be elements of a group G and suppose $z = [x, y]$ commutes with both x and y. Then $[x^i, y^j] = z^{ij}$ for all integers i and j* [6, *p.* 19, *Theorem* 2.2.2].

(vii) *Let ψ be a p'-automorphism of a p-group P which induces the identity on $P/\Phi(P)$. Then ψ is the identity automorphism of P* [6, *p.* 174, *Theorem* 5.1.4].

(viii) *If A is a p'-group of automorphisms of the p-group P with p odd which acts trivially on $\Omega_1(P)$, then $A = 1$* [6, *p.* 184, *Theorem* 5.3.10].

(ix) *If A is a noncyclic abelian normal subgroup of a p-group P then there exists $V \leqslant A$ such that $Z_p \times Z_p \cong V \lhd P$.*

(x) *If P is a p-group of class at most 2 with p odd then $\Omega_1(P)$ has exponent p* [6, *p.* 183, *Lemma* 5.3.9 (i)].

(xi) *(Thompson) Let P be a p-group. Then P contains a characteristic subgroup Q with class at most two with the property that any p'-automorphism of P that acts trivially on Q must also act trivially on P (Q is called a critical subgroup of P)* [6, *p.* 185, *Theorem* 5.3.11].

The proof of (ix) is easy. These results are absolutely basic and will mostly be quoted without reference.

LEMMA 1.2 (H. Bender). *Let M and L be maximal local subgroups of a group G. Suppose L and M are $\mathcal{N}$-constrained and $F(L)$ contains a subgroup S such that $C_{F(L)}(S) \leqslant S \leqslant M$. Let $\pi = \pi(F(L))$. Then $O^{\pi}(F(M)) \cap L = 1$. If $|\pi| > 1$ and $O^{\pi}(F(M)) = 1$ then $M = L$.*

Proof. The proof of [3, p. 166, Theorem 1.7] can be revised to yield a proof of this lemma. Note that if $F(G) \neq 1$ then $M = L = G$.

THEOREM 1.3 [7, Theorem B]. *Let G be a group and $A \in \mathscr{A}(2, G)$. If $V \leqslant G$ is nilpotent and normalized by A then AV is nilpotent.*

THEOREM 1.4 [7, Prop. 1]. *Let V be a nontrivial finite dimensional $GF(q)$-vector space. Let $A \leqslant GL(V)$ be a nilpotent q'-group of class at most two. Then $|A| < |V|$.*

THEOREM 1.5 (H. Bender). *Let G be a group and $I \in \mathcal{N}\mathcal{I}(G)$. Then I contains every nilpotent subgroup of G which it normalizes.*

Proof (See Also [1]). Assume false. Let B be a nilpotent subgroup of G of minimal order such that B is normalized by I but that $B \nleqslant I$. It is easily seen that B is a p-group for some prime p and that $[B, O_{p'}(I)] \neq 1$.

If $C_B(O_p(I)) \neq B$ then minimality of B implies that $C_B(O_p(I)) \leqslant I$ and hence $[C_B(O_p(I)), O_{p'}(I)] = 1$. The $P \times Q$ Lemma implies that $[B, O_{p'}(I)] = 1$, a contradiction. Hence $[B, O_p(I)] = 1$.

Let B_0 be a critical subgroup of B. If $B_0 \neq B$ then $B_0 \leqslant I$ and $[B_0, O_{p'}(I)] = 1$. Lemma 1.1 (xi) implies that $[B, O_{p'}(I)] = 1$, again a contradiction. We deduce that $B_0 = B$ and that $\mathrm{cl}(B) \leqslant 2$.

Let A be a member of $\mathscr{A}(2, G)$ that is contained in I. Theorem 1.3 implies that AB is nilpotent and hence $[B, O_{p'}(A)] = 1$. Since $[B, O_p(I)] = 1$ we deduce that $[B, A] = 1$. Since $\mathrm{cl}(A) \leqslant 2$ and $\mathrm{cl}(B) \leqslant 2$ we see that $\mathrm{cl}(AB) \leqslant 2$ and as $A \in \mathscr{A}(2, G)$ we deduce that $B \leqslant A \leqslant I$, a contradiction.

LEMMA 1.6. *Let G be an $\mathcal{N}$-constrained group, $p \in \pi(F(G))$, $I \in \mathcal{N}\mathcal{I}(G)$, and $O_p(I) \leqslant P \in \mathrm{Syl}_p(G)$. Then*

(i) $O_p(I) \in \mathrm{Syl}_p(C(O_{p'}(F(G))))$,

(ii) $O_p(I) \lhd P$, *and*

(iii) $\mathcal{N}\mathcal{I}(G)$ *is a single conjugacy class of subgroups.*

Proof. By Theorem 1.5, I is a maximal nilpotent subgroup of G that contains $F(G)$. Now see [9].

LEMMA 1.7. *Let p be an odd prime, k and n natural numbers with $n \geqslant 2$. If $k^p \equiv 1 \bmod p^n$ then $k \equiv 1 \bmod p^{n-1}$.*

Proof. See [10, p. 50, Theorem 4.5 (b)].

2. PRELIMINARY LEMMAS

LEMMA 2.1. *Let q be a prime and V a finite dimensional $GF(q)$-vector space. Let $A \leqslant GL(V)$ be an irreducible nilpotent q'-group of class at most two. Let $a \in A$ be of prime order r with $C_V(a) \neq 0$. Then a has r distinct conjugates $a_0 = a, a_1, \ldots, a_{r-1}$.*

Set $C = C_A(a)$ then $C \lhd A$, $C = C_A(a_i)$, $0 \leqslant i < r$, and $|A : C| = r$. Set $V_i = C_V(a_i)$, $0 \leqslant i < r$. Then $V = V_0 \oplus \cdots \oplus V_{r-1}$, each V_i is C-invariant and the V_i's are permuted by A.

Proof. As A is irreducible, $Z(A)$ is regular and cyclic. Hence $a \notin Z(A)$.

Since A has class at most two, $\langle Z(A), a\rangle \lhd A$. Let z be a generator of $\Omega_1(O_r(Z(A)))$. Then $Z_r \times Z_r \cong \langle z, a\rangle \lhd A$ and $|A:C| = r$. We see that $C \lhd A$ and $C = C_A(\langle z, a\rangle) = C_A(a_i)$, $0 \leqslant i < r$. For $0 \leqslant i < r$ set $a_i = z^i a$, so the a_i's are the distinct conjugates of a.

Let n be maximal such that $\langle V_0 \dots V_n\rangle = V_0 \oplus \cdots \oplus V_n$. If $n < r-1$, choose $v_{n+1} \in \langle V_0, \dots, V_n\rangle \cap V_{n+1}^{\#}$. Then $v_{n+1} = v_0 + \cdots + v_n$ with $v_i \in V_i$ $0 \leqslant i < n$. So $v_0 + \cdots + v_n = v_{n+1} = v_{n+1}^{a_{n+1}} = v_0^{a_{n+1}} + \cdots + v_n^{a_{n+1}}$. As $a_{n+1} \in C = C_A(a_i)$, we have $v_i^{a_{n+1}} \in V_i = C_V(a_i)$. So as $\langle V_0, \dots, V_n\rangle = V_0 \oplus \cdots \oplus V_n$, we have $v_i = v_i^{a_{n+1}}$ for $0 \leqslant i \leqslant n$. Since $v_{n+1} \neq 0$ it is possible to choose j, $0 \leqslant j \leqslant n$, such that $v_j \neq 0$. Then $\langle a_{n+1}, a_j\rangle \leqslant C_{\langle z, a\rangle}(v_j)$ and $j < n+1$ implies $\langle a_{n+1}, a_j\rangle = \langle z, a\rangle$. Hence $v_j \in C_V(z) = 0$. This is a contradiction since $Z(A)$ is regular on V. So $n = r-1$ and $\langle V_0, \dots, V_{r-1}\rangle = V_0 \oplus \cdots \oplus V_{r-1}$. Clearly the V_i's are permuted by A and irreducibility implies $V = V_0 \oplus \cdots \oplus V_{r-1}$.

LEMMA 2.2. *Assume the hypothesis and notation of Lemma* 2.1. *Suppose also that a is chosen to maximize* $|C_V(a)|$. *Set* $C_i = C_A(V_i)$, $0 \leqslant i < r$.

Then $C_i \lhd C$ *and* $C_i = \langle a_i\rangle \cong Z_r$, $0 \leqslant i < r$. *Also* $C_i \cap C_j = 1$ *if* $i \neq j$.

Proof. Fix i. By Lemma 2.1, $|A:C| = r$. If $C_i \nleqslant C$ then $A = CC_i$. However, C_i and C normalize V_i, contradicting the irreducibility of A. So $C_i \leqslant C$ and $C_i = C_A(V_i) \lhd C$ as C normalizes V_i.

Since $\mathrm{cl}(A) \leqslant 2$ we see that $[C_i, C_i] \leqslant Z(A)$. However, $Z(A)$ is regular so $[C_i, C_i] = 1$. Now let $i \neq j$ and let C_i act on V_j. Maximality of $|C_V(a)|$ implies that $C_i \cap C_j = 1$ and that C_i is regular on V_j. As C_i is abelian we deduce C_i is cyclic.

Let $g \in O_r(A) - C$, $C_i = \langle c_i\rangle$ and $V_i^g = V_j$. Then $i \neq j$ as $g \notin C$. Also $C_i^g = C_j$. Let $c_j = c_i^g$, so $C_j = \langle c_j\rangle$. As $g^r \in C \lhd C_i$ and $\mathrm{cl}(A) \leqslant 2$, Lemma 1.1 (vi) yields $[c_i^r, g] = [c_i, g]^r = [c_i, g^r] \in C_i$ and $[c_j^r, g] \in C_j$. Now

$$1 = [c_i, g, g] = [c_i^{-1} g^{-1} c_i g, g] = [c_i^{-1} c_j, g] = [c_i, g]^{-1}[c_j, g].$$

Hence $[c_i, g] = [c_j g]$ and $[c_i^r, g] = [c_j^r, g] \in C_i \cap C_j = 1$. So $\langle c_i^r\rangle \lhd A = \langle C, g\rangle$. As $V_i \leqslant C_V(c_i^r)$, irreducibility of A forces $C_V(c_i^r) = V$. Hence $c_i^r = 1$.

LEMMA 2.3. *Suppose* $1 < R \lhd Q \lhd P$ *with* $\mathrm{cl}(P) \leqslant 2$, $|P:Q| = p$, Q/R *and* $Z(P)$ *cyclic, and* $Z(P) \cap R = 1$. *Then* $Q = Z(P) \times R$ *and* $R \cong Z_p$.

Proof. Let $g \in P - Q$ so $P = Q\langle g\rangle$. As Q/R is cyclic, $[Q, Q] \leqslant R$ and $[Q, Q] \lhd P$ but $Z(P) \cap [Q, Q] \leqslant Z(P) \cap R = 1$ and this forces Q to be abelian. Also $Z(P) \leqslant Q$ since P cannot be abelian.

Next we show $[P, P] = \Omega_1(Z(P)) \cong Z_p$. For, as $Z(P)$ is cyclic $\Omega_1(Z(P)) \cong Z_p$, and then $\mathrm{cl}(P) \leqslant 2$ implies that $[P, P] \leqslant Z(P)$. It is only necessary to show that if $a \in Q$ and $b \in P - Q$ then $[a, b]$ has order at most p. Since $b^p \in Q$ and Q is abelian, $[a, b]^p = [a, b^p] = 1$.

Now define a map $\psi : Q \to [P, P]$ by $x\psi = [x, g]$. Since $\mathrm{cl}(P) = 2$, ψ is an epimorphism. It is easy to see that $\ker \psi = Z(P)$ and $Q/Z(P) \cong [P, P] \cong Z_p$. Now as $Z(P) \cap R = 1$, we have $Q = Z(P) \times R$ and $R \cong Z_p$.

LEMMA 2.4. *Let* $G = \langle X \rangle$. *If* $[x, y] \in Z(G)$ *for all* $x, y \in X$, *then* $\mathrm{cl}(G) \leqslant 2$.

Proof. The proof is easy using [6, Theorem 2.2.1].

LEMMA 2.5. *Let* P *be a* p*-group*, $p \neq 2$. *Let* $C \leqslant P$ *with* $C \cong Z_{p^n} \times Z_p$, $n \geqslant 1$. *If* $g \in N_P(C)$ *with* $g^p \in C_P(C)$ *and* $[g, \Omega_1(C)] = 1$ *then* $\mathrm{cl}(\langle C, g \rangle) \leqslant 2$.

Proof. We have $C = \langle a, b \rangle$ with $a^{p^n} = b^p = [a, b] = 1$. If $n = 1$ then $C = \Omega_1(C)$ and by assumption, $\langle C, g \rangle$ is abelian.

So assume $n \geqslant 2$. As $[g, \Omega_1(C)] = 1$ we have $b^g = b$. We have $i, j \in \mathbb{Z}$ such that $a^g = a^i b^j$. By induction, for all $k \in \mathbb{N}$, $a^{g^k} = a^{i^k} b^{j(1 + i + \cdots + i^{k-1})}$. Since $g^p \in C_P(C)$ and $C = \langle a \rangle \times \langle b \rangle$, this implies $i^p \equiv 1 \mod p^n$. So by Lemma 1.7, $i \equiv 1 \mod p^{n-1}$.

By Lemma 2.4 and since C is an abelian normal subgroup of $\langle C, g \rangle$, it is sufficient to show that $[a, g, g] = [b, g, g] = 1$. Now $[b, g] = 1$ as $b^p = 1$ and $[g, \Omega_1(C)] = 1$. We can find $k \in \mathbb{Z}$ such that $i = 1 + kp^{n-1}$. Then $[a, g] = a^{-1} g^{-1} a g = a^{-1} a^{1 + kp^{n-1}} b^j = a^{kp^{n-1}} b^j$. So $[a, g]^g = (a^g)^{kp^{n-1}} (b^g)^j = (a^{1 + kp^{n-1}} b^j)^{kp^{n-1}} b^j = a^{kp^{n-1} + k^2 p^{2n-2}} b^{jkp^{n-1}} b^j$. As $n \geqslant 2$, $b^{p^{n-1}} = 1$. If $2n - 2 < n$ then $n < 2$, a contradiction. So $a^{p^{2n-2}} = 1$. Hence $[a, g]^g = a^{kp^{n-1}} b^j = [a, g]$ so $[a, g, g] = 1$ and $\mathrm{cl}(\langle C, g \rangle) \leqslant 2$.

LEMMA 2.6. *Let* X *and* Y *be groups.*

Then $\mathscr{A}(2, X \times Y) = \{A \times B \mid A \in \mathscr{A}(2, X), B \in \mathscr{A}(2, Y)\}$.

Proof. Trivial.

LEMMA 2.7. *Let* P *be a* p*-group and* $A \in \mathscr{A}(2, P)$. *If* $Z_p \times Z_p \cong V \lhd P$, *then* $V \leqslant A$.

Proof. Since $[V, A] \neq V$ we have $[V, A] \lesssim Z_p$. Hence $[V, A] \leqslant Z(VA)$. A Sylow p-subgroup of $\mathrm{Aut}(V)$ is abelian so $[V, [A, A]] = 1$ and $[A, A] \leqslant Z(VA)$. By Lemma 2.4, $\mathrm{cl(VA)} \leqslant 2$. The definition of $\mathscr{A}(2, P)$ implies $V \leqslant A$.

LEMMA 2.8. *Let* P *be a* p*-group and suppose every member of* $\mathscr{A}(2, P)$ *is either cyclic*, D_8, *or* Q_8. *Then* P *is quasicyclic.*

Proof. If false then P contains a noncyclic abelian normal subgroup $V \cong Z_p \times Z_p$. Let $A \in \mathscr{A}(2, P)$. By Lemma 2.7, $V \leqslant A$ so $A \cong D_8$ and $P > A$. Now $A = \langle a, b \rangle$, where $a^4 = b^2 = 1$ and $a^b = a^{-1}$. Let $g \in N_P(A) - A$. The group A has only two elements of order four, a and a^{-1}. So by replacing g with gb if necessary we may suppose $a^g = a$. Set $B = \langle a, g \rangle$. Then $|B| > |\langle a \rangle| = 4$ so $|B| \geqslant 8$. Since B is abelian, $A \in \mathscr{A}(2, P)$ and $|A| = 8$ we deduce that $|B| = |A|$ and $B \in \mathscr{A}(2, P)$. By Lemma 2.7, $V \leqslant B$ so B cannot be cyclic, D_8, or Q_8, a contradiction.

LEMMA 2.9. *Let G be a group, $A \in \mathscr{A}(2, G)$ and $A \leqslant I \in \mathscr{N}\mathscr{I}(G)$. Let $p \in \pi(A)$ and suppose $O_p(A)$ is cyclic, then $O_p(I)$ is quasicyclic. Also if $O_p(A) \cong Z_p$ then $O_p(I) \cong Z_p$.*

Proof. Without loss of generality $G = I$ and by Lemma 2.6 we can assume that $I = O_p(I)$. If I is not quasicyclic then Lemma 1.1 (ii) implies that we can find $V \lhd I$ such that $V \cong Z_p \times Z_p$. Lemma 2.7 implies $V \leqslant O_p(A)$, a contradiction. The last assertion is obvious.

LEMMA 2.10. *Let G be a group, $I, J \in \mathscr{N}\mathscr{I}(G)$ and $p \in \pi d(2, G)$. If $|O_p(I)| = p$ then $|O_p(J)| = p$.*

Proof. Let $A \leqslant I$ and $B \leqslant J$ with $A, B \in \mathscr{A}(2, G)$. Then $p = |O_p(A)| = |d(2, G)|_p = |O_p(B)|$. The previous lemma gives $O_p(J) \cong Z_p$.

LEMMA 2.11. *Let P be a p-group and $Q \lhd P$. If Q possesses noncyclic abelian normal subgroups then either one of them is normal in P or $Q \cong D_8$.*

Proof. If $p \neq 2$ then by Lemma 1.1 (v), $\Omega_1(Z_2(Q))$ is noncyclic. Now let V be a subgroup of $\Omega_1(Z_2(Q))$ of order p^2 normal in P. As $p \neq 2$, $\Omega_1(Z_2(Q))$ has exponent p so $V \cong Z_p \times Z_p$.

Hence we may assume $p = 2$ and a counterexample has been chosen with $|Q|$ minimal. By Lemma 1.1(ix) there exists V such that $Z_2 \times Z_2 \cong V \lhd Q$. Now let D be a cyclic normal subgroup of P contained in Q of maximal order. So $C_Q(D) \lhd P$. If $D \neq C_Q(D)$ choose $E \lhd P$ with $D < E \leqslant C_Q(D)$ and $|E:D| = 2$. Then E is abelian and the choice of D implies E is noncyclic, a contradiction. So $D = C_Q(D)$.

If $|D| = 2$ then $D \leqslant Z(Q)$ but then $D = C_Q(D) = Q \geqslant V$, a contradiction. If $|D| = 4$ then as $Z_2 \times Z_2 \cong V \lhd Q$ and $C_Q(D) = D$ we see that $|Q:D| = 2$ and $Q \cong D_8$, a contradiction. So $|D| > 4$. Let $D = \langle d \rangle$ and $|D| = 2^m$ so $m > 2$. As $V \lhd Q$, $V \leqslant C_Q(d^2) \lhd P$. If $C_Q(d^2) < Q$ then by minimality of Q, $C_Q(d^2) \cong D_8$ so d^2 has order 2 and $|D| = 4$, a contradiction. So $d^2 \in Z(Q)$. There is exactly one nontrivial automorphism of D that centralizes d^2. As $C_Q(D) = D$ we have $|Q:D| = 2$ and $Q = DV$.

Since $D = C_Q(D)$, $1 \neq [D, V] \leqslant D \cap V \lneqq Z_2$ so $[D, V] = D \cap V = \langle d^{2^{m-1}} \rangle$.

Let $v \in V - D$ so $[d, v] = d^{2^{m-1}}$. Suppose $g \in Q - \{v, d^{2^{m-1}}\}$ has order 2. Then as $Q = \langle d \rangle \langle v \rangle \rhd \langle d \rangle$, for some $i \in \mathbb{N}$, $g = d^i v$. Now $1 = g^2 = d^i v\, d^i v = d^i (v\, dv)^i = d^i (dd^{-1} v\, dv)^i = d^i (dd^{2^{m-1}})^i = d^{2i + 2^{m-1} i}$, so $2i + 2^{m-1} i = 2^m a$ for some $a \in \mathbb{N}$. As $m > 2$, $i(1 + 2^{m-2}) = 2^{m-1} a$ and $2^{m-1} \mid i$ and $g = d^{2^{m-1}} v$. Hence $V = \langle v, d^{2^{m-1}}, d^{2^{m-1}} v \rangle = \Omega_1(Q) \lhd P$, a contradiction.

3. Local Analysis

In this section G is a finite group all of whose local subgroups are $\mathcal{N}$-constrained, and we let $\pi = \pi d(2, G)$. The aim is to show that all nilpotent injectors of G are conjugate. If $\pi = \{p\}$ then it is easy to see $\mathcal{N}\mathcal{I}(G) = \mathrm{Syl}_p(G)$. Hence, by Sylow's Theorem, we may assume $|\pi| > 1$.

Lemma 3.1. *If $A \in \mathcal{A}(2, G) \cup \mathcal{N}\mathcal{I}(G)$ and $A \leqslant L \in \Lambda(G)$ then $\pi(A) = \pi(F(L)) = \pi(AF(L)) = \pi$.*

Proof. By Theorem 1.3 or 1.5, $AF(L)$ is nilpotent. Since L is a maximal local subgroup and is $\mathcal{N}$-constrained, $C_L(F(L)) \leqslant F(L)$ and the result follows.

Theorem 3.2. *Let $I \in \mathcal{N}\mathcal{I}(G)$ then I is contained in a unique maximal local subgroup.*

Proof. Suppose $I \leqslant L, M \in \Lambda(G)$. Then by Theorem 1.5 $F(L) \leqslant I \leqslant M$ and $F(M) \leqslant I \leqslant L$. By Lemma 3.1 $\pi = \pi(F(L)) = \pi(F(M))$. Therefore $O^{\pi}(F(M)) = 1$. Now apply Lemma 1.2 with $S = F(L)$.

Lemma 3.3. *Let $I \in \mathcal{N}\mathcal{I}(G)$, $I \leqslant L \in \Lambda(G)$ and $Z(I) \leqslant M \in \Lambda(G)$. If $O^{\pi}(F(M)) = 1$ then $M = L$.*

Moreover, if $A \in \mathcal{A}(2, I)$ then L is the unique maximal local subgroup containing A.

Proof. Let $p \in \pi$ and let $Z = Z(I) = O_p(Z) \times O_{p'}(Z)$ act on $O_p(M)$. By Theorem 3.2, $C(O_p(Z))$, $C(O_{p'}(Z)) \leqslant L$. Theorem 1.5 implies $F(L) \leqslant I$. Since L is $\mathcal{N}$-constrained, $Z \leqslant F(L)$ hence $O_{p'}(Z) \leqslant O_{p'}(L)$. We deduce $[C_{O_p(M)}(O_p(Z)), O_{p'}(Z)] \leqslant O_p(M) \cap O_{p'}(L) = 1$. The $P \times Q$ Lemma implies $[O_p(M), O_{p'}(Z)] = 1$. Hence $O_p(M) \leqslant L$. Since $O^{\pi}(F(M)) = 1$, we see that $F(M) \leqslant L$.

If $p \in \pi(F(L)) - \pi(F(M))$ then as $O_p(Z) \leqslant O_p(L)$, we have $[O_p(Z), F(M)] \leqslant O_p(L) \cap F(M) = 1$. Since M is $\mathcal{N}$-constrained $O_p(Z) \leqslant F(M)$, a contradiction. Hence $\pi = \pi(F(L)) \subseteq \pi(F(M))$ and as $|\pi| > 1$, Lemma 1.2 (with $S = F(M)$ and the roles of M and L interchanged) implies $M = L$.

Now we prove the second assertion. Suppose $A \leqslant N \in \Lambda(G)$. Lemma 3.1

implies $O^{\pi}(F(N))=1$. It follows trivially from the definition of A that $Z(I)\leqslant A$, so from what we have just proved, $N=L$.

LEMMA 3.4. *Let $I\in\mathcal{N}\mathcal{I}(G)$, $I\leqslant L\in\Lambda(G)$ and $p\in\pi$. If $O_p(I)\in \mathrm{Syl}_p(L)$ then I is conjugate to every other member of $\mathcal{N}\mathcal{I}(G)$.*

Proof. Let $O_p(I)\leqslant P\in \mathrm{Syl}_p(G)$. Theorem 3.2 implies $N(O_p(I))\leqslant L$ hence $N_P(O_p(I))\leqslant O_p(I)$. This implies $O_p(I)=P$ so $O_p(I)\in \mathrm{Syl}_p(G)$.

Let $J\in\mathcal{N}\mathcal{I}(G)$ and $J\leqslant M\in\Lambda(G)$. By Sylow's Theorem we may suppose $O_p(J)\leqslant O_p(I)$. So $Z(I)\leqslant N(O_p(J))$ and Theorem 3.2 gives $Z(I)\leqslant M$. By Lemma 3.1, $\pi(F(M))=\pi$ so Lemma 3.3 implies $M=L$. Hence $I, J\in\mathcal{N}\mathcal{I}(L)$ and since L is $\mathcal{N}$-constrained, I and J are conjugate by Lemma 1.6 (iii).

LEMMA 3.5. *Let $I\in\mathcal{N}\mathcal{I}(G)$ and $I\leqslant L\in\Lambda(G)$. Suppose for each prime p, $F(L)$ does not contain a noncyclic abelian subgroup that is normal in a Sylow p-subgroup of L. Then I is conjugate to every other member of $\mathcal{N}\mathcal{I}(G)$.*

Proof. By Lemmas 1.1 (ii) and 2.11, $O(F(L))$ is cyclic. If $O_2(L)$ contains no noncyclic abelian normal subgroup then $F(L)$ is quasicyclic. If $O_2(L)$ contains a noncyclic abelian normal subgroup then by Lemma 2.11, $O_2(L)\cong D_8$. In either case $F(L)$ is quasicyclic and $O_2(L)$ is not isomorphic to $Z_2\times Z_2$.

Suppose the lemma is false and let $J\in\mathcal{N}\mathcal{I}(G)$ with I not conjugate to J. Let $p=\max\pi$ (so that $p>2$), $J\leqslant M\in\Lambda(G)$ and $O_p(I)\leqslant P\in \mathrm{Syl}_p(L)$. First we will show that $\pi=\{2,3\}$ and $O_2(L)$ is nonabelian. If $[P, O_{p'}(F(L))]=1$ then by Lemma 1.6 (i), $P=O_p(I)$ contradicting Lemma 3.4. So choose $q\in\pi-\{p\}$ such that $[P, O_q(L)]\neq 1$. If $O_q(L)$ is cyclic then $O_q(L)/\Phi(O_q(L))\cong Z_q$, hence Lemma 1.1 (vii) and the fact that $p>q$ imply $[P, O_q(L)]=1$, a contradiction. So $O_q(L)$ is noncyclic. Hence $O_q(L)$ is quasicyclic but noncyclic. This implies $q=2$ and $O_2(L)/\Phi(O_2(L))\cong Z_2\times Z_2$. Then Lemma 1.1 (vii) implies $p=3$. As $p=\max\pi$, $\pi=\{2,3\}$. The only noncyclic quasicyclic abelian 2-group is $Z_2\times Z_2$. As $O_2(L)$ is not isomorphic to $Z_2\times Z_2$, $O_2(L)$ is nonabelian.

Now let $O_2(I)\leqslant S\in \mathrm{Syl}_2(L)$. As $O(F(L))=O_3(L)$ is cyclic, Lemma 1.6 (i) implies $|S: O_2(I)|\leqslant 2$. So $1<[S,S]\lhd I$ and $N_G(S)\leqslant N_G([S,S])\leqslant L$ by Theorem 3.2. Hence $S\in \mathrm{Syl}_2(G)$. By Sylow's Theorem, we may suppose $O_2(J)\leqslant S$. Set $T=O_2(I)\cap O_2(J)$. Since $O_2(L)$ is a nonabelian subgroup of $O_2(I)$ we see that $|O_2(I)|>2$ so by Lemma 2.10, $|O_2(J)|>2$. Hence as $|S: O_2(I)|\leqslant 2$, we have $1<T\lhd O_2(J)$. Since J is nilpotent $T\lhd J$. So $\langle J, Z(I)\rangle\leqslant N(T)$. Theorem 3.2 gives $Z(I)\leqslant N(T)\leqslant M$. Lemmas 3.1 and 3.3 imply $M=L$ and the result follows by Lemma 1.6 (iii).

LEMMA 3.6. *Suppose the following hold*:

(1) $p \in \pi$,

(2) $A \in \mathscr{A}(2, G)$,

(3) $Z(A) \leqslant B < A$, $|A:B| = p$,

(4) $q \in \pi'$,

(5) Q *is a q-group of class at most two normalized by* B,

(6) B *acts irreducibly on* $Q/\Phi(Q)$, *and*

(7) $Z(A)$ *acts faithfully on* Q.

Set $\hat{B} = B/C_B(Q)$. *Then* $C_B(Q)$ *is a p-group,* $O(\hat{B})$ *is cyclic and* $O_2(\hat{B})$ *is cyclic,* D_8 *or* Q_8.

Proof. Set $V = Q/\Phi(Q)$. Since $O_{p'}(B) = O_{p'}(A)$, (7) implies $C_B(Q)$ is a p-group. If $\hat{B}$ is regular on V then $\hat{B}$ is quasicyclic so as $\mathrm{cl}(B) \leqslant 2$ we see that $O_2(\hat{B})$ is cyclic, D_8 or Q_8. Hence we may assume $\hat{B}$ is not regular on V.

Since $A \in \mathscr{A}(2, G)$, (5) implies $|QC_B(Q)| < |A| = p|B|$, hence,

$$|V| < p|\hat{B}|. \tag{α}$$

Since $\hat{B}$ is not regular on V, choose $a \in \hat{B}^{\#}$ so as to maximize $|C_V(a)|$. Without loss of generality we can assume a has prime order r. Let $a_0, a_1, ..., a_{r-1}, C, V_0, V_1, ..., V_{r-1}, C_0, C_1, ..., C_{r-1}$ be as in Lemmas 2.1 and 2.2. As $V = V_0 \oplus \cdots \oplus V_{r-1}$ and $|\hat{B}:C| = r$, (α) becomes

$$|V_0|^r < pr\,|C|. \tag{β}$$

By Theorem 1.4, $|C:C_0| < |V_0|$ and by Lemma 2.2, $C_0 \cong Z_r$ so

$$\frac{1}{r}\,|C| \leqslant |V_0| - 1. \tag{γ}$$

Set $m = (1/r^2)\,|C|$. By Lemma 2.2, $C_0 \cap C_1 = 1$ and $Z_r \times Z_r \cong C_0 C_1 \leqslant C$. Hence r^2 divides $|C|$ and m is an integer. Observe that (3) and (7) imply that $|Z(A)|$ divides $|C|$. Substitute $|C| = mr^2$ in (β) and (γ) and then eliminate $|V_0|$ to obtain

$$(mr+1)^r < pmr^3. \tag{δ}$$

If $p = r$ then, by (δ), $(mr)^r < mr^4$ so $r = 2$ or 3 and

$$m < 4 \qquad \text{if} \quad r = 2$$

$$m^2 < 3 \qquad \text{if} \quad r = 3.$$

Since $|\pi| \geqslant 2$ and $|Z(A)|$ divides $|C|$ we see that m is divisible by a prime that is not equal to r. This implies $r=2$ and $m=3$. Now substitute these values in (δ) to obtain $(3 \cdot 2+1)^2 < 2 \cdot 3 \cdot 2^3$, i.e., $49 < 48$, a contradiction. Hence $p \neq r$.

Recall that $|Z(A)|$ divides $|C|$. Then p divides $|C|$ and as $p \neq r$, p divides m. So by (δ), $(mr)^r < m^2 r^3$. This forces $r=2$ and together with (δ) forces $m < 2p$. Since m is an integer divisible by p we see that $m=p$ and $|C|=4p$. Hence

$$C \cong Z_2 \times Z_2 \times Z_p.$$

Since $|\hat{B}:C|=2$ we deduce that $O(\hat{B}) \cong Z_p$. Then $O_2(\hat{B}) \cong D_8$ since C is a centralizer.

LEMMA 3.7. *Let $I \leqslant L \in A(G)$, $p \in \pi$ and $Z_p \times Z_p \cong V \lhd I \in \mathcal{NI}(G)$ with $V \leqslant F(L)$. Let $O_p(I) \leqslant P \in \mathrm{Syl}_p(L)$ and $v \in V^{\#}$. If $C_G(v) \nleqslant L$ then*

(I) *$O_{p'}(I)$ is quasicyclic, and*

(II) *$1 < [P, P] \lhd I$.*

Furthermore if $p \neq 2$ then

(III) *$\mathrm{cl}(O_p(I)) = 2$, and*

(IV) *$\Omega_1(O_p(I))$ is isomorphic to a subgroup of the extraspecial group of order p^3 and exponent p.*

Proof. Let $C_G(v) \leqslant M \in A(G)$. Since by hypothesis $C_G(v)$ is not contained in L, we know that $L \neq M$. First we will show:

(*) If $A \in \mathscr{A}(2, I)$ then $O_2(O_{p'}(A))$ is cyclic, D_8, or Q_8; $O(O_{p'}(A))$ is cyclic; and $Z(A)$ is cyclic. Moreover if $p \neq 2$ then $O_p(C_A(V)) \cong Z(O_p(A)) \times Z_p$ and $V = \Omega_1(C_{O_p(A)}(V))$.

Let $B = C_A(V)$. By Lemmas 2.6 and 2.7, $V \leqslant A$. Hence $V \lhd A$ and $|A:B| = 1$ or p. Set $S = C_{F(L)}(V) \geqslant V$. Then $C_{F(L)}(S) \leqslant S \leqslant M$ and $B \leqslant M$. By Lemma 1.2, there exists a prime $q \in \pi'$ such that $O_q(M) \neq 1$ so by Lemma 3.1, $A \nleqslant M$ and so $|A:B| = p$. Let Q be a minimal B-invariant subgroup of $O_q(M)$. Then Q is elementary abelian and B is irreducible on Q. By Lemma 1.2, $Q \cap L = 1$ and Lemma 3.3 implies that $Z(A)$ is faithful on Q.

Now set $\hat{B} = B/C_B(Q)$ and apply Lemma 3.6. We see that $C_B(Q)$ is a p-group, $O_2(\hat{B})$ cyclic, D_8, or Q_8 and $O(\hat{B})$ cyclic. So as $C_B(Q)$ is a p-group and $O_{p'}(A) = O_{p'}(B)$, $O_2(O_{p'}(A))$ is cyclic, D_8 or Q_8 and $O(O_{p'}(A))$ is cyclic. As $Z(A) \lesssim Z(\hat{B})$, $Z(A)$ is cyclic.

To prove the last statement, assume $p \neq 2$. If $C_B(Q) = 1$ then $O_p(B)$ is cyclic contradicting $Z_p \times Z_p \cong V \leqslant B$, so $1 < C_B(Q) \lhd B \lhd A$ and $|A:B| = p$.

Also as $Z(A)$ is faithful on Q, $Z(A) \cap C_B(Q) = 1$. Now apply Lemma 2.3 with $C_B(Q)$ in place of R, $O_p(B)$ in place of Q and $O_p(A)$ in place of P. We obtain $O_p(B) = Z(O_p(A)) \times C_B(Q)$ with $C_B(Q) \cong Z_p$. As $Z(A)$ is cyclic, $V = \Omega_1(O_p(C_A(V))$.

Proof of (I). Let $A \in \mathscr{A}(2, I)$ and $S \in \mathscr{A}(2, O_{p'}(I))$. By Lemma 2.6, $SO_p(A) \in \mathscr{A}(2, I)$, so by $(*)$ $O(S)$ is cyclic and $O_2(S)$ is cyclic, D_8 or Q_8. Since this is true for all members of $\mathscr{A}(2, O_{p'}(I))$, Lemmas 2.6 and 2.8 imply $O(O_{p'}(I))$ is cyclic and $O_2(O_{p'}(I))$ is quasicyclic.

Proof of (II). Let $A \in \mathscr{A}(2, I)$. Since $V \leqslant O_p(A)$ and $Z(A)$ is cyclic we see that $O_p(A)$ is nonabelian and $[P, P] \neq 1$. By Lemma 1.6, $O_p(I) = P \cap C(O_{p'}(F(L))) \lhd P$. So it is sufficient to show $[P, P] \leqslant C(O_{p'}(F(L)))$. By Theorem 1.5, $F(L) \leqslant I$. By (I), $O(O_{p'}(F(L)))$ is cyclic and is therefore centralized by $[P, P]$. By (I), $O_2(O_{p'}(\mathrm{I}))$ is quasicyclic and so $O_2(O_{p'}(F(L)))$ is quasicyclic and $O_2(O_{p'}(F(L)))/\Phi(O_2(O_{p'}(F(L)))) \lesssim Z_2 \times Z_2$. This implies $[P, P]$ centralizes $O_2(O_{p'}(F(L)))$. Hence $[P, P]$ centralizes $O_{p'}(F(L))$.

Now fix $A \in \mathscr{A}(2, I)$ and assume $p \neq 2$.

Proof of (III). It suffices to show $O_p(A) = O_p(I)$. Set $C = C_{O_p(A)}(V)$ and $D = C_{O_p(I)}(V)$. By $(*)$, $V \nleqslant Z(A)$ so $O_p(I) = O_p(A)D$ since a Sylow p-subgroup of $\mathrm{Aut}(V)$ has order p. Hence it suffices to show $C = D$. If $D > C$ choose $g \in N_D(C) - C$ such that $g^p \in C_D(C)$. By $(*)$, $V = \Omega_1(C)$ and there exists an integer n such that $C \cong Z_{p^n} \times Z_p$. Since $g \in C(V)$, Lemma 2.5 gives $\mathrm{cl}(\langle C, g \rangle) \leqslant 2$. As $|O_p(A) : C| = p$ we see that $|\langle C, g \rangle| \geqslant |O_p(A)|$ and Lemma 2.6 implies $O_{p'}(A) \langle C, g \rangle \in \mathscr{A}(2, I)$. So $Z_p \times Z_p \cong V \leqslant Z(O_{p'}(A) \langle C, g \rangle)$ but by $(*)$, $Z(O_{p'}(A) \langle C, g \rangle)$ is cyclic, a contradiction. So $C = D$.

Proof of (IV). Since $A \in \mathscr{A}(2, I)$, Lemma 2.6 and (III) imply $O_p(A) = O_p(I)$. Then by $(*)$, $O_p(A)$ contains a subgroup $E \cong Z_{p^n} \times Z_p$ of index p. As $p \neq 2$ and $\mathrm{cl}(O_p(I)) = 2$, Lemma 1.1 (x) implies $\Omega_1(O_p(I))$ has exponent p. This implies $E \cap \Omega_1(O_p(I)) = \Omega_1(E)$. Hence $\Omega_1(E)$ has index at most p in $\Omega_1(O_p(I))$ and as $\Omega_1(E) \cong Z_p \times Z_p$, either (IV) follows or $\Omega_1(O_p(I))$ is elementary abelian of order p^3.

If $\Omega_1(O_p(I))$ is abelian then, using $O_p(I) = O_p(A)$, we see that $\Omega_1(O_p(I)) \leqslant \Omega_1(C_{O_p(A)}(V))$. The last assertion in $(*)$ implies $\Omega_1(O_p(I)) \leqslant V$ so $\Omega_1(O_p(I))$ cannot be abelian of order p^3.

THEOREM 3.8. *Let $p \in \pi$, $I \in \mathscr{N}\mathscr{I}(G)$, $I \leqslant L \in A(G)$ and $O_p(I) \leqslant P \in \mathrm{Syl}_p(L)$. If $O_p(L)$ contains a noncyclic abelian subgroup which is normal in P then $P \in \mathrm{Syl}_p(G)$.*

Proof. It suffices to show $N(P) \leqslant L$. Let $Z_p \times Z_p \cong V \leqslant O_p(L)$ with $V \lhd P$. Note that since $O_p(L) \leqslant O_p(I) \leqslant P$ and since I is nilpotent we also have $V \lhd I$. If for some $v \in V^{\#}$, $C_G(v) \nleqslant L$ then by Lemma 3.7, $1 < [P, P] \lhd I$ and by Theorem 3.2, $N(P) \leqslant N([P, P]) \leqslant L$.

So assume $C_G(v) \leqslant L$ for all $v \in V^{\#}$. Let $g \in N(P)$. Then $V \leqslant P = P^g \leqslant L^g$. Lemma 1.1 (iv) gives $O_{p'}(F(L^g)) = \langle C_{O_{p'}(F(L^g))}(v) \mid v \in V^{\#} \rangle \leqslant L$. Now $P \in \mathrm{Syl}_p(L^g)$ so $O_p(L^g) \leqslant P \leqslant L$ and $F(L^g) \leqslant L$. Since $\pi(F(L)) = \pi(F(L^g))$, Lemma 1.2 (with L^g taking the role of L, L the role of M and $S = F(L^g)$) implies $L = L^g$ so $g \in N(O_p(L)) = L$.

THEOREM. *$\mathcal{N}\mathcal{I}(G)$ is a single conjugacy class of subgroups.*

Proof. Suppose false. Let $I, J \in \mathcal{N}\mathcal{I}(G)$ with I not conjugate to J. Let $I \leqslant L \in \Lambda(G)$ and $J \leqslant M \in \Lambda(G)$. By Theorem 1.5, $F(L) \leqslant I$ and $F(M) \leqslant J$. By Lemma 3.5 we can find $p, q \in \pi$ and V, W, P, Q such that $Z_p \times Z_p \cong V \leqslant F(L) \leqslant I$, $V \lhd P \in \mathrm{Syl}_p(L)$, $Z_q \times Z_q \cong W \leqslant F(M) \leqslant J$ and $W \lhd Q \in \mathrm{Syl}_q(M)$. By Theorem 3.8, P and Q are Sylow subgroups of G.

Assume that $C_G(v) \leqslant L$ for all $v \in V^{\#}$. By Sylow's Theorem we may suppose $O_p(J) \leqslant P$. By Lemma 2.10, $|O_p(J)| > p$ so $1 < C_{O_p(J)}(V) \lhd J$; hence $\langle J, V \rangle \leqslant N(C_{O_p(J)}(V))$, and Theorem 3.2 gives $V \leqslant M$. The assumption on centralizers and Lemma 1.1 (iv) imply $O_{p'}(F(M)) \leqslant L$. Also $O_p(M) \leqslant O_p(J) \leqslant P \leqslant L$ so $F(M) \leqslant L$. Lemmas 3.1, 1.2, and the fact that $|\pi| > 1$ imply $L = M$ and Lemma 1.6 (iii) provides a contradiction.

Hence we may suppose there exists $v \in V^{\#}$ such that $C_G(v) \nleqslant L$. By the same argument we may also suppose there exists $w \in W^{\#}$ such that $C_G(w) \nleqslant M$.

If $p = q$ then by Sylow's Theorem we may suppose $P = Q$. By Lemma 3.7, $1 < [P, P] \lhd I$ and $[P, P] \lhd J$. Hence $\langle I, J \rangle \leqslant N([P, P])$ and Theorem 3.2 implies that $L = M$. Lemma 1.6(iii) supplies a contradiction.

We may assume that $p > q$. Let $r = \max \pi$ and $O_r(I) \leqslant R \in \mathrm{Syl}_r(L)$. If $[R, O_{r'}(F(L))] = 1$ then Lemma 1.6 (i) gives $O_r(I) = R \in \mathrm{Syl}_r(L)$ contradicting Lemma 3.4. Choose $t \in \pi - \{r\}$ such that $[R, O_t(L)] \neq 1$. As $F(L) \leqslant I$ and $p \neq 2$, Lemma 3.7 (IV) gives $\Omega_1(O_p(L))/\Phi(\Omega_1(O_p(L))) \lesssim Z_p \times Z_p$. If $r > p$ then Lemma 1.1 (vii), (viii) and the structure of $\mathrm{Aut}(Z_p \times Z_p)$ and $\mathrm{Aut}(Z_p)$ imply that $[R, O_p(L)] = 1$. Since $r = \max \pi$ we deduce that $t \neq p$. By Lemma 3.7 (I), $O(O_{p'}(F(L)))$ is cyclic and $r = \max \pi$ implies $t = 2$ and $O_2(L)$ is noncyclic. By Lemma 3.7 (I), $O_2(L)/\Phi(O_2(L)) \lesssim Z_2 \times Z_2$ and $[R, O_2(L)] \neq 1$ implies $r = 3$. As $r = \max \pi$ we deduce that $\pi = \{2, 3\}$, $p = 3$ and $q = 2$.

Since $O_3(I) \leqslant P \in \mathrm{Syl}_3(L)$ and $O_2(L)/\Phi(O_2(L)) \lesssim Z_2 \times Z_2$, Lemma 1.6 (i) implies $|P : O_3(I)| \leqslant 3$. Then $O_3(I) \lhd P$. By Theorem 3.8, $P \in \mathrm{Syl}_3(G)$ so we may assume $O_3(J) \leqslant P$. Set $T = O_3(J) \cap O_3(I)$. Since $O_3(I) \lhd P$ it follows that $T \lhd O_3(J)$ and then that $T \lhd J$ since J is nilpotent. Since $Z_3 \times Z_3 \cong V \leqslant I$, Lemma 2.10 implies $|O_3(J)| > 3$. Since $|P : O_3(I)| \leqslant 3$ we deduce that $T \neq 1$. Hence $1 < T \lhd J$. So $\langle Z(I), J \rangle \leqslant N(T)$ and Theorem 3.2 gives $Z(I) \leqslant M$. Lemmas 3.1 and 3.3 give $M = L$. The conclusion now follows from Lemma 1.6 (iii).

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